Compositions of permutations and algorithmic reducibilities

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Abstract

For *wtt*-reducibility or an arbitrary reducibility stronger then *wtt*-reducibility, there exists a set $F \leq_T 0'$ such that the set $\{f \mid f \text{ is a permutation of } \omega \text{ and the graph of } f \text{ reduce to } F\}$ is not closed under composition.

Let A be an arbitrary set such that $A \leq_T 0'$. There exists a permutation f such that the graph of f is 2-c.e. computable and the graph of f^2 is not *wtt*-reducible to A.

All necessary definitions can be found in [3, 4]. Algorithmic properties of permutations on natural numbers are studied in many papers. A lot of authors studied groups $G_d = \{f \mid f \text{ is a permutation of } \omega \text{ and } f \leq_T d\}$ where d is a Turing degree. It turns out that any group G_d characterizes the degree d, i.e two such groups are isomorphic iff the corresponding degrees coincide and embedding on such groups is equivalent to Turing reducibility on degrees (see [2]). In [2] A. S. Morozov has formulated the following problem: is it possible to extend these results to other reducibilities? In our paper we demonstrate that for some wide class of algorithmic reducibilities there is negative answer for this question.

Notation. We fix some effective bijective numbering of pairs of integers i.e. three computable functions $\langle \cdot, \cdot \rangle : \omega^2 \to \omega$; $(\cdot)_1, (\cdot)_2 : \omega \to \omega$ such that $\langle (x)_1, (x)_2 \rangle = x$, $(\langle x_1, x_2 \rangle)_1 = x_1$, $(\langle x_1, x_2 \rangle)_2 = x_2$.
Let us define function $\langle x_1, \ldots, x_n \rangle = \langle \ldots \langle \langle x_1, x_2 \rangle, x_3 \rangle, \ldots, x_n \rangle$ for $n > 2$.

Let D_n denote a finite set with a canonical index n, i.e. $n = \sum_{j \in D_n} 2^j$.
Let $f : \omega \to \omega$; Γ_f denotes the graph of f, i.e. the set $\{\langle x, y\rangle \mid f(x) = y\}$.

Definition. A set A is *weak truth-table reducible* to B ($A \leq_{wtt} B$) if there exist a natural number z and a computable function g such that
1) $\chi_A = \{z\}^B$
2) the set $D_{g(x)}$ contains all numbers whose membership or nonmembership in B are used in the computation of $\{z\}^B(x)$.
We say that pair $(\{z\}, g)$ reduces A to B.

Definition. A finite function $f : \omega \to \omega$ is a *path* if
$\Gamma_f = \{\langle m_0, m_1\rangle, \langle m_1, m_2\rangle, \ldots \langle m_{k-1}, m_k\rangle\}$, where $m_i \neq m_j$ for $i \neq j$.
Let $b(f) = m_0$ denote the first element of the path and $e(f) = m_k$ denote the last element of the path.

Definition. Let $D \subseteq \omega$,
we define field$(D) = \{(m)_1 \mid m \in D\} \cup \{(m)_2 \mid m \in D\}$. For a function f, field(f) =field(Γ_f) =dom$(f)\cup$range(f).

We shall consider reducibility of a function f to a set A as reducibility of the graph of f to A. In this case the following theorem holds.

Theorem 1. Let $\preceq$ be wtt-reducibility or an arbitrary reducibility stronger then wtt-reducibility (i.e. $A \preceq B$ implies $A \leq_{wtt} B$). There exists a set $F \leq_T 0'$ such that the set

$$\{f \mid f \text{ is a permutation of } \omega \text{ and } \Gamma_f \preceq F\}$$

is not closed under composition.

Remark. It is well known that the reducibilities $\leq_1$, $\leq_m$, $\leq_{btt}$, $\leq_{tt}$ are stronger then wtt-reducibility. So, the proposition of the theorem holds for them.

Proof. We shall construct a permutation f with the following properties: $f \leq_T 0'$ and $\Gamma_{f^2} \not\leq_{wtt} \Gamma_f$. Obviously, the graph of f will be the required set F. We construct the permutation f in stages by a finite extension $0'$-oracle construction, so that

$$\Gamma_{f_0} \subseteq \Gamma_{f_1} \subseteq \ldots \subseteq \bigcup_{i\in\omega} \Gamma_{f_i} = \Gamma_f.$$

Every f_i will be a path. It suffices to meet, for each stage s, the following requirement: the pair of functions $(\{(s)_1\}, \{(s)_2\})$ does not reduce Γ_{f^2} to Γ_f.

Stage $s = 0$. Define $\Gamma_{f_0} = \{\langle 0, 1\rangle\}$.

Stage $s > 0$. Define $p_s = \mu n(n \notin \text{field}(f_s))$, $q_s = \mu n(n \notin \text{field}(f_s) \cup \{p_s\})$, $\Gamma_{f'_s} = \Gamma_{f_s} \cup \{\langle p_s, b(f_s)\rangle, \langle e(f_s), q_s\rangle\}$ and $\beta_s = \mu n(n \notin \text{field}(f'_s))$. We denote $(s)_1$ by z and $\{(s)_2\}$ by g. Let us consider all possible cases for $g(\langle e(f'_s), \beta_s\rangle)$.

Case 1. $g(\langle e(f'_s), \beta_s\rangle)$ is undefined. Then we define $f_{s+1} = f_s$ and turn to the next stage.

Case 2. $g(\langle e(f'_s), \beta_s\rangle)$ is defined. Then we define $l_s = g(\langle e(f'_s), \beta_s\rangle)$ and $D^+_{l_s} = D_{l_s} \cap \Gamma_{f'_s}$, $D^-_{l_s} = D_{l_s} \setminus D^+_{l_s}$. We shall extend the path f'_s to the path f''_s with the following condition: the graph of any extension of f''_s contains no element of $D^-_{l_s}$. For any $m = \langle u, v\rangle \in D^-_{l_s}$, we carry out the following procedure. A number c will be called a *new element* if $c \notin \text{field}(D^-_{l_s})$ and c does not belong to the field of the constructed path and $c \neq \beta_s$. Let c be a new element and a be the first element of the constructed path. If $u \neq \beta_s$ and u does not belong to the field of the path, then we add the pairs (u, c) and (c, a) to the path. If $u = \beta_s$, v does not belong to the field of the path and $v \neq u$, then we add the pairs (v, c) and (c, a) to the constructed path. In other cases we do nothing. We denote the constructed path by f''_s.

Let us consider the function $\{z\}$. If $\{z\}^{\Gamma_{f''_s}}(\langle e(f''_s, \beta_s\rangle)$ is defined and equal to 1, then we add the pairs $\langle e(f''_s), c\rangle, \langle c, d\rangle, \langle d, \beta_s\rangle$ to the path, where c and d are sequentially chosen new elements. In other cases we add the pairs $\langle e(f''_s), c\rangle, \langle c, \beta_s\rangle$, where c is a new element. We denote the constructed path by f_{s+1}.

It is clear that $\Gamma_f = \bigcup_{i\in\omega} \Gamma_{f_i}$ is the graph of a permutation, and $\Gamma_f \leq_T 0'$.

The theorem follows from the lemmas below.

Lemma 1. If at Stage s Case 2 holds, then for all m, $m \in D^-_{l_s}$ implies $m \notin \Gamma_f$.

Proof. Let $m \in D^-_{l_s}$ and $m = \langle u, v\rangle$. Let us consider all possible cases.

By the construction, the permutation f is an infinite cycle, so $u = v$ implies $m \notin \Gamma_f$. Consider the case $u \in \text{dom}(f''_s)$. If $u \in \text{dom}(f'_s)$, then $\langle u, v\rangle \notin \Gamma_f$ by the definition of $D^-_{l_s}$. If we include u into the field of f''_s when we constructing f''_s, then $\langle u, c\rangle \in \Gamma_{f''_s}$ and c is a new element, i.e. $c \neq v$. If

$u = e(f''_s)$, then $\langle u, c\rangle \in \Gamma_{f_{s+1}} \subset \Gamma_f$, where c is a new element, i.e. $\langle u, v\rangle \notin \Gamma_f$.

If $u = \beta_s$ and $u \neq v$, then $v \in \mathrm{dom}(f_{s+1})$ and $\beta_s = e(f_{s+1})$, so f is an infinite cycle; hence $\langle u, v\rangle \notin \Gamma_f$.

Lemma 2. $\Gamma_{f^2} \not\leq_{wtt} \Gamma_f$.

Proof. Assume that $\Gamma_{f^2} \leq_{wtt} \Gamma_f$, i.e. there exists a pair of functions $(\{z\}, \{p\})$ that reduces Γ_{f^2} to Γ_f. Let us consider a stage $s = \langle z, p\rangle$. Note that $\{z\}^{\Gamma_f}(\langle e(f''_s), \beta_s\rangle) = \{z\}^{\Gamma_{f''_s}}(\langle e(f''_s), \beta_s\rangle)$. It is clear that $e(f''_s) = e(f'_s)$, i.e. $\{p\}(\langle e(f''_s), \beta_s\rangle) = l_s$ by the construction. If $m \in D_{l_s}$ and $m \in \Gamma_{f''_s}$, then $m \in \Gamma_f$, since $\Gamma_{f''_s} \subset \Gamma_f$. If $m \in D_{l_s}$ and $m \notin \Gamma_{f''_s}$, then $m \in D^-_{l_s}$ and according Lemma 1, we have $m \notin \Gamma_f$. So therefore $D_{l_s} \cap \Gamma_{f''_s} = D_{l_s} \cap \Gamma_f$, i.e.

$$\{z\}^{\Gamma_f}(\langle e(f''_s), \beta_s\rangle) = \{z\}^{\Gamma_{f''_s}}(\langle e(f''_s), \beta_s\rangle). \tag{1}$$

By the assumption, $\chi_{\Gamma_{f^2}} = \{z\}^{\Gamma_f}$. There are two possible cases.

Case 1. $\{z\}^{\Gamma_f}(\langle e(f''_s), \beta_s\rangle) = 1$. The equality (1) implies

$$\{z\}^{\Gamma_{f''_s}}(\langle e(f''_s), \beta_s\rangle) = 1.$$

By the construction of f_{s+1}, the pairs $\langle e(f''_s), c\rangle$ and $\langle c, d\rangle$ are in the graph of f, where d is a new element. We obtain that $f^2(e(f''_s)) = d \neq \beta_s$, $\langle e(f''_s), \beta_s\rangle \notin \Gamma_{f^2}$, i.e.,

$$0 = \chi_{\Gamma_{f^2}}(\langle e(f''_s), \beta_s\rangle) = \{z\}^{\Gamma_f}(\langle e(f''_s), \beta_s\rangle) = 1.$$

Contradiction.

Case 2. $\{z\}^{\Gamma_f}(\langle e(f''_s), \beta_s\rangle) = 0$ handled by analogy with Case 1.

Lemma 2 and Theorem 1 are proved.

Let us consider the arithmetical hierarchy.

Remark. The set

$$\{f \mid f \text{ is a permutation and } \Gamma_f \in \Sigma^0_n\}$$

is the group of all $0^{(n-1)}$ computable permutations.

We denote $\mathrm{dom}(\{i\}^A_s)$ by $W^A_{i,s}$.

Theorem 2. The set

$$\{f \mid f \text{ is a permutation and } \Gamma_f \in \Pi^0_n\}$$

is not closed under composition for any $n \geq 1$.

Proof. We shall construct a permutation f with the following properties: $\Gamma_f \in \Pi^0_n$ and $\Gamma_{f^2} \notin \Pi^0_n$. We construct the permutation f in stages in the following way, we computable enumerate in $0^{(n-1)}$ the complement of Γ_f and meet the requirement R_i: complement of Γ_{f^2} does not coincide with $W_i^{0^{(n-1)}}$ for any i. Requirement R_i will be met on a fixed pair $\langle a_i, c_i\rangle$. At every stage s we construct a finite path f_s, and a finite set B_s that contains all pairs of elements of the fieldf_s, which are not in Γ_{f_s}. The required permutation f will be the limit of the sequence $\{f_s\}_{s\in\omega}$, and $B_0 \subset B_1 \subset \ldots \subset \bigcup_{i\in\omega} B_s = \omega \setminus \Gamma_f$.

Stage $s = 0$. Define $\Gamma_{f_0} = \{\langle 0,1\rangle\}$, $B_0 = \{\langle 1,0\rangle\}$.

Stage $s > 0$. Let s be in a row i, i.e. $s = \langle i,j\rangle$. A number c will be called a *new element* if c does not belong to the field of the constructed path. Let p_s and q_s be distinct new elements. Define $\Gamma_{f'_s} = \Gamma_{f_s} \cup \{\langle p_s, b(f_s)\rangle, \langle e(f_s), q_s\rangle\}$. Let us consider the function $\{i\}$.

Case 1. We have not considered the function $\{i\}$ at the previous stages. It means that $\forall s'(s' < s \to (s')_1 \neq i)$. Let a_i,b_i,c_i be distinct new elements. Define $\Gamma_{f_{s+1}} = \Gamma_{f'_s} \cup \{\langle e(f'_s), a_i\rangle, \langle a_i, b_i\rangle, \langle b_i, c_i\rangle\}$.

Case 2. We have considered the function $\{i\}$ at the previous stages but we have not met the requirement R_i and $\langle c_i, a_i\rangle \in W_{i,s}^{0^{(n-1)}}$. Let n_{i_1},n_{i_2},n_{i_3},n_{i_4} be distinct new elements. Define

$$\Gamma_{f_{s+1}} = \left(\Gamma_{f'_s} \setminus \{\left\langle f'^{-1}_s(a_i), a_i\right\rangle, \langle a_i, b_i\rangle, \langle b_i, c_i\rangle, \langle c_i, f'_s(c_i)\rangle\}\right) \bigcup$$
$$\{\left\langle f'^{-1}_s(a_i), n_{i_1}\right\rangle, \langle n_{i_1}, c_i\rangle, \langle c_i, n_{i_2}\rangle, \langle n_{i_2}, a_i\rangle,$$
$$\langle a_i, n_{i_3}\rangle, \langle n_{i_3}, b_i\rangle, \langle b_i, n_{i_4}\rangle, \langle n_{i_4}, f'_s(c_i)\rangle\}$$

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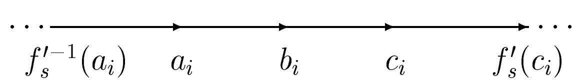
We have changed f'_s:



to the path f_{s+1}:

$$\dots \; f_s'^{-1}(a_i) \longrightarrow n_{i_1} \longrightarrow c_i \longrightarrow n_{i_2} \longrightarrow a_i \longrightarrow n_{i_3} \longrightarrow b_i \longrightarrow n_{i_4} \longrightarrow f_s'(c_i) \; \dots$$

We have obtained $f^2_{s+1}(c_i) = a_i$, and $\langle c_i, a_i \rangle \in W^{0^{(n-1)}}_{i,s}$, i.e. we met the requirement R_i.

In other cases we define $f_{s+1} = f'_s$.

Then we define

$$B_{s+1} = \{\langle c, d\rangle \mid (c \in \text{field} f_{s+1})\&(d \in \text{field} f_{s+1})\&(\langle c, d\rangle \notin \Gamma_{f_{s+1}})\}.$$

By the construction, it is clear that there exists the limit of the sequence $\{f_s\}_{s\in\omega}$, we denote it as f. It is clear that f is a permutation. Our construction is computable in $0^{(n-1)}$ and $B_0 \subset B_1 \subset \dots \subset \bigcup_{i\in\omega} B_s = \omega \setminus \Gamma_f$, hence $\omega \setminus \Gamma_f$ is c.e. in $0^{(n-1)}$ set. We have met all requirements R_i.

The theorem is proved.

Let us consider permutations whose graphs are limit-computable.

Definition. A function f is *limit-computable* if $f(x) = \lim_s g(s, x)$ for some total computable function g. Let $f(x) = \lim_s g(s, x)$ and g is a total computable function. Define $k_{f,g} =\mid \{s \mid g(s + 1, x) \neq g(s, x)\} \mid$. A function f is *n-c.e.* if $\forall x k_{f,g}(x) \leq n$. A function f is *ω-c.e.* if $\forall x k_{f,g}(x) \leq h(x)$ for some total computable function h. A set A is *limit computable*, *n-c.e.*, *ω-c.e.*, if its characteristic function is limit computable, n-c.e., ω-c.e. correspondingly (we suppose that $\chi_A(x) = \lim_s A(s, x)$ and $A(0, x) = 0$).

Theorem 3. Let A be an arbitrary limit computable set. There exists a permutation f such that Γ_f is 2-c.e. and $\Gamma_{f^2} \not\leq_{wtt} A$.

Proof. We construct the permutation f in stages. It suffices to meet, for each s (s in a row i), the following requirement R_i : the pair $(\{(i)_1\}, \{(i)_2\})$ does not reduce Γ_{f^2} to A. At every stage s we construct a finite path f_s. The required permutation will be the limit of the sequence $\{f_s\}_{s\in\omega}$.

The requirement R_i will be met on a fixed pair $\langle a_i, c_i\rangle$. We obtain that if $\{(i)_1\}^A(\langle a_i, c_i\rangle)$ use in the computation questions to the oracle A only about elements of $D_{\{(i)_2\}(\langle a_i, c_i\rangle)}$ then $\{(i)_1\}^A(\langle a_i, c_i\rangle) \neq \chi_{\Gamma_{f^2}}(\langle a_i, c_i\rangle)$. The set A

is limit computable, let $A(s,x)$ be a total computable function such that $\chi_A = \lim_s A(s,x)$. Let us denote $\{x \mid A(s,x)=1\}$ by A_s.

Stage $s=0$. Define $\Gamma_{f_0} = \{\langle 0,1\rangle\}$.

Stage $s>0$. Let s be in a row i, i.e. $s=\langle i,j\rangle$. A number c will be called a *new element* if c does not belong to the field of the constructed path. Let p_s and q_s be distinct new elements. Define $\Gamma_{f'_s} = \Gamma_{f_s} \cup \{\langle p_s, b(f_s)\rangle, \langle e(f_s), q_s\rangle\}$. Let us consider the pair of the functions $(\{(i)_1\},\{(i)_2\})$.

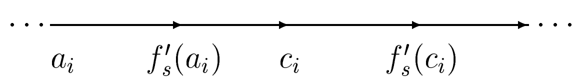
Case 1. We have not considered the pair $(\{(i)_1\},\{(i)_2\})$ at the previous stages. It means that $\forall s'(s'<s \rightarrow (s')_1 \neq i)$. Let a_i,b_i,c_i be distinct new elements. Define $\Gamma_{f_{s+1}} = \Gamma_{f'_s} \cup \{\langle e(f'_s), a_i\rangle, \langle a_i, b_i\rangle, \langle b_i, c_i\rangle\}$.

Case 2. We have considered the pair $(\{(i)_1\},\{(i)_2\})$ at the previous stages and $\{(i)_2\}(\langle a_i,c_i\rangle)$ converges and $\{(i)_1\}_s^{A_s}(\langle a_i,c_i\rangle) = \chi_{\Gamma_{f_s^2}}(\langle a_i,c_i\rangle)$ and $\{(i)_1\}_s^{A_s}(\langle a_i,c_i\rangle)$ use in the computation questions to the oracle A_s only about elements of $D_{\{(i)_2\}(\langle a_i,c_i\rangle)}$.

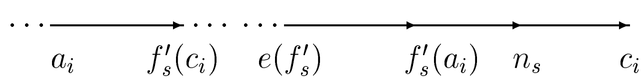
Let us consider all possible cases for $\{(i)_1\}^{A_s}(\langle a_i,c_i\rangle)$.

Case 2.1. $\{(i)_1\}^{A_s}(\langle a_i,c_i\rangle) = 1 = \chi_{\Gamma_{f_s^2}}(\langle a_i,c_i\rangle)$. Let n_s be a new element. Then we define $\Gamma_{f_{s+1}} = \left(\Gamma_{f'_s} \setminus \{\langle a_i, f'_s(a_i)\rangle, \langle f'_s(a_i), c_i\rangle, \langle c_i, f'_s(c_i)\rangle\}\right) \cup \{\langle a_i, f'_s(c_i)\rangle, \langle e(f'_s), f'_s(a_i)\rangle, \langle f'_s(a_i), n_s\rangle, \langle n_s, c_i\rangle\}$.

We have changed the path f'_s:



to the path f_{s+1}:



Case 2.2. $\{(i)_1\}^{A_s}(\langle a_i,c_i\rangle) = 0 = \chi_{\Gamma_{f_s^2}}(\langle a_i,c_i\rangle)$. Let n_{s_1},n_{s_2} be distinct new elements. Then we define $\Gamma_{f_{s+1}} = \left(\Gamma_{f'_s} \setminus \{\langle a_i, f'_s(a_i)\rangle, \langle f'^{-1}_s(c_i), c_i\rangle, \langle c_i, f'_s(c_i)\rangle\}\right) \cup \{\langle a_i, n_{s_1}\rangle, \langle n_{s_1}, c_i\rangle, \langle c_i, n_{s_2}\rangle, \langle n_{s_2}, f'_s(a_i)\rangle, \langle f'^{-1}_s(c_i), f'_s(c_i)\rangle\}$.

We have changed the path f'_s:

$$\cdots \xrightarrow{} \cdots \quad \cdots \xrightarrow{} \xrightarrow{} \cdots$$
$$a_i \quad f'_s(a_i) \quad f'^{-1}_s(c_i) \quad c_i \quad f'_s(c_i)$$

to the path f_{s+1}:

$$\cdots \xrightarrow{} \xrightarrow{} \xrightarrow{} \xrightarrow{} \cdots \quad \cdots \xrightarrow{} \cdots \quad \cdots$$
$$a_i \quad n_{s_1} \quad c_i \quad n_{s_2} \quad f'_s(a_i) \quad f'^{-1}_s(c_i) \quad f'_s(c_i)$$

We have obtained $\{(i)_1\}^{A_s}(\langle a_i, c_i\rangle) \neq \chi_{\Gamma_{f^2_{s+1}}}(\langle a_i, c_i\rangle)$.

Let us demonstrate that there exists the limit of the sequence $\{f_s\}_{s\in\omega}$. It suffices to test pairs containing a_i,c_i for some i. Let us fix an arbitrary i. We changed pairs containing a_i,c_i only when the condition of Case 2 was true. The set A is limit computable and the set $D_{\{(i)_2\}(\langle a_i,c_i\rangle)}$ is finite, hence the set $A_s \cap D_{\{(i)_2\}(\langle a_i,c_i\rangle)}$ have not changed after some stage s'. After the stage s' the pairs containing a_i,c_i will be fixed, i.e. $f_{s'+1}(a_i) = f(a_i)$, $f^{-1}_{s'+1}(a_i) = f^{-1}(a_i)$, $f_{s'+1}(c_i) = f(c_i)$, $f^{-1}_{s'+1}(c_i) = f^{-1}(c_i)$. We have demonstrated that there exists limit of the sequence $\{f_s\}_{s\in\omega}$. We have met the requirement R_i at the stage s'. So we obtain that $\Gamma_{f^2} \not\leq_{wtt} A$. By the construction it is clear that Γ_f is 2-c.e.. The theorem is proved.

Corollary 1. Let A be an arbitrary set such that $A \leq_T 0'$. There exists a permutation f such that Γ_f is 2-c.e. but $\Gamma_{f^2} \not\leq_{wtt} A$.

Proof. A set A is limit computable iff $A \leq_T 0'$, by Limit Lemma [4].

Definition. Let n be a natural number, we say that $A \leq_{ntt} B$ if there exists computable total function f with the following conditions: for all x the norm of tt-condition $f(x)$ is less or equal to n, and $x \in A \leftrightarrow (f(x)$ is satisfied by $B)$.

Remark. If a set A is 2-c.e., then $A \leq_{2tt} \emptyset'$.

Proof. Let us consider the c.e. set $M = \{\langle i, x, a\rangle \mid \exists t \mid \{s \mid s \leq t \& (|A(s,x) \neq A(s+1,x)\}| = i) \& A(t,x) = a\}$. It is clear that $A \leq_{2tt} M$, since $x \in A \leftrightarrow (\langle 1, x, 1\rangle \in M \& \langle 2, x, 0\rangle \notin M)$.

Corollary 2. There exists a permutation f such that $\Gamma_f \leq_{2tt} \emptyset'$ and $\Gamma_{f^2} \not\leq_{wtt} \emptyset'$.

Corollary 3. The set $\{f \mid f \text{ is a permutation and } \Gamma_f \leq_{tt} \emptyset'\}$ is not closed under composition.

Corollary 4. The set $\{f \mid f \text{ is a permutation and } \Gamma_f \leq_{wtt} \emptyset'\}$ is not closed under composition.

Taking into account the result by H.G. Garstens [1] that a set A is ω-c.e. iff $A \leq_{tt} \emptyset'$, we obtain that the set

$$\{f \mid f \text{ is a permutation and } \Gamma_f \text{ is } \omega\text{-c.e. }\}$$

is not closed under composition.

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